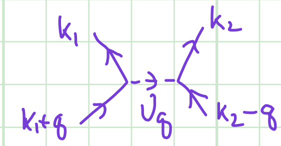
Plan: (1) Collective modes of the Fermi sea

(2) Imaginary freq. poles + spontaneous symmetry breaking

Last time:

Hamiltonian for the interacting Fermi sea

$$H = \sum_{k\sigma} \underbrace{\left(\frac{k^2}{2m} - \mu\right)}_{\xi_k} c^\dagger_{k\sigma} c_{k\sigma} + \sum_{\substack{k_1 k_2 q \\ \sigma_1 \sigma_2}} V_{q\sigma_1\sigma_2}\, c^\dagger_{k_1\sigma_1} c_{k_1+q\,\sigma_1} c^\dagger_{k_2\sigma_2} c_{k_2-q\,\sigma_2}$$



where $V_{q\sigma_1\sigma_2}$ is the Fourier transform of the inter-fermionic potential energy.

For the case of electrostatic interactions
e.g. for electrons in solids:

$$V_{q\sigma_1\sigma_2} = \frac{4\pi e^2}{q^2} \qquad [\text{note, } \sigma\text{-independent}]$$

For the case of point-contact interactions
e.g. for ultracold atoms, "hard spheres"

$$V_{q\sigma_1\sigma_2} = \begin{cases} \frac{ma}{4\pi} \equiv U & \sigma_1 \neq \sigma_2 \quad [a_s \text{ is the scattering length}] \\ 0 & \sigma_1 = \sigma_2 \quad [\text{no scattering of like spins due to Pauli}] \end{cases}$$

Our goal is to describe the collective modes of $\mathcal{H}$

$\Rightarrow$ The collective modes appear as poles in appropriate susceptibilities [aka correlation functions].

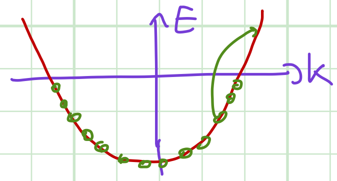
$\Rightarrow$ This is just like the electron appears as the pole in

$$G(q,\omega) \equiv -i \int dt\, e^{i\omega t} \langle T\, c_q(t)\, c_q^\dagger(0)\rangle = \frac{1}{\omega - \varepsilon_q + i\delta_q}$$

$\omega = \varepsilon_q \Rightarrow$ electron pole $\Rightarrow$ prescribes the energy to an electron with momentum q

Before discussing various susceptibilities let's talk about elementary excitations of the Fermi sea.

Starting with Fermi sea in ground state the elementary excitations consist of taking an electron from below the Fermi surface + Installing above the Fermi surface



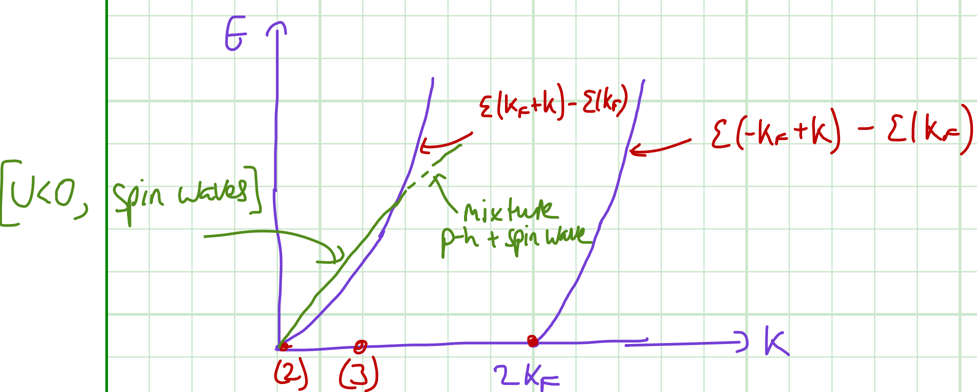
$c_k^\dagger c_p |FS\rangle$ where $|k| > k_F > |p|$

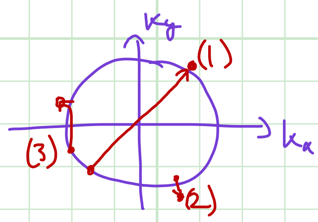
These are, unsurprisingly called the particle-hole excitations

Let us plot the momentum – Energy diagram for these excitations

Momentum: $k-p$ [WRT the ground state]

Energy : $\varepsilon_k - \varepsilon_p$





$$\frac{\cancel{k_F^2}+2k_Fk+k^2}{2m} - \frac{\cancel{k_F^2}}{2m} = \varepsilon(k) + v_F k \quad (\checkmark)$$

[agrees with Mahan

Susceptibilities:

What kind of collective modes can we have in a Fermi gas?

(1) Density waves $\Rightarrow -i\langle \rho(x,t)\, \rho(x',0)\rangle \equiv \chi_{\rho\rho}(x-x',t)$

(2) Spin waves [Magnetization] $\Rightarrow -i\langle S_\alpha(x,t)\, S_\beta(x',0)\rangle \equiv \chi_{\alpha\beta}(x-x',t)$

(3) Pair-density waves $\Rightarrow -i\langle \Delta^\dagger(x,t)\, \Delta(x,0)\rangle \equiv \chi_{\Delta\Delta}(x-x',t)$ [This one needs slight refinement]

Let us start with the spin susceptibility. The tools we use for it are easily generalized for the other susceptibilities. At this point we have two main avenues:

(1) Equation of motion

$\langle i\dot{S}_\alpha(x,t)\rangle = \langle [S_\alpha(x,t), H]\rangle$

$\Rightarrow$ Break up four operator terms $\langle c^\dagger c\, c^\dagger c\rangle \rightarrow \langle c^\dagger c\rangle\langle c^\dagger c\rangle$

This is the Random phase in RPA. We will save this approach for homework.

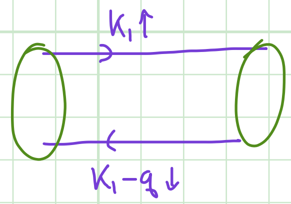
(2) use resummation of diagrams

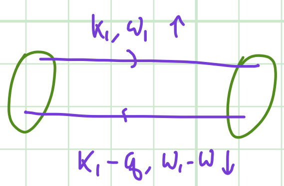
By arbitrary choice, let me focus on χ_{+-}.

dagger

$$\chi_{+-}(q,\omega) = -i\int dt\, e^{i\omega t}\langle S_+(q,t)\, S_-^{+}(+q,0)\rangle = -i\int dt\, e^{i\omega t}\left\langle\left(\sum_{k_1} c_{k_1\uparrow}^{+}(t)\, c_{k_1-q\downarrow}(t)\right)\left(\sum_{k_2} c_{k_2\downarrow}^{+}(0)\, c_{k_2+q\uparrow}(0)\right)\right\rangle$$

The non interacting part

$$\chi_0(q,t) = S^+(q,t)$$  $$S^-(-q,0) = -i\int dk_1\; G^{(0)}_{k_1\uparrow}(t)\, G^{(0)}_{k_1-q\downarrow}(-t)$$

$$\chi_0(q,\omega) = S^+(q,\omega)$$  $$S^-(-q,-\omega) = -i\int dk_1 \frac{d\omega_1}{(2\pi)}\; G^{(0)}_{k_1\uparrow}(\omega_1)\, G^{(0)}_{k_1-q\downarrow}(\omega_1-\omega)$$

$$= \int dk_1 \frac{d\omega_1}{2\pi}\left[\frac{1}{\omega_1 - \xi_{k_1} + i\delta_{k_1}}\;\frac{1}{\omega_1-\omega-\xi_{k_1-q}+i\delta_{k_1-q}}\right] \qquad \delta_k = \delta\, \mathrm{sign}(\xi_k)$$

Strategy: Perform the $d\omega_1$ integral first, then perform the dk_1 integral

poles: $\omega_1 = \xi_{k_1} - i\delta_{k_1}$

$\omega_1 = \omega + \xi_{k_1-q} - i\,\delta_{k_1-q}$

contour integral non-zero only if one pole in upper-half plane + the other in lower half-plane

			δ_{k_1}	δ_{k_1-q}	
four cases: (1)	$\xi_{k_1}>0$	$\xi_{k_1-q}>0$	+ (l)	+ (l)	$\Rightarrow \oint = 0$
(2)	$\xi_{k_1}>0$	$\xi_{k_1-q}<0$	+ (l)	− (u)	
(3)	$\xi_{k_1}<0$	$\xi_{k_1-q}>0$	− (u)	+ (l)	
(4)	$\xi_{k_1}<0$	$\xi_{k_1-q}<0$	− (u)	− (u)	$\oint \to 0$

$$\chi_0 = -i\,\frac{2\pi i}{2\pi}\int dk_1\left[\overset{(2)}{\frac{\theta(\xi_{k_1})\left[1-\theta(\xi_{k_1-q})\right]}{\omega+\xi_{k_1-q}-\xi_{k_1}}} + \overset{(3)}{\frac{\theta(\xi_{k_1-q})\left[1-\theta(\xi_{k_1})\right]}{-\omega+\xi_{k_1}-\xi_{k_1-q}}}\right] = \int dk_1\, \frac{(1-n_F(\xi_{k_1}))\,n_F(\xi_{k_1-q}) - (1-n_F(\xi_{k_1-q}))\,n_F(\xi_{k_1})}{\omega+\xi_{k_1-q}+\xi_{k_1}}$$

$$= \int dk_1\, \frac{n_F(\xi_{k_1-q/2}) - n_F(\xi_{k_1+q/2})}{\omega+\xi_{k_1-q/2}-\xi_{k_1+q/2}} \equiv \text{Lindhard function}$$

Surprisingly, this integral can be performed analytically. It is a bit of a pain and the resulting answer is not particularly useful except for numerics. As proof, let me write the answer:

$$\chi_0(q,\omega) = \frac{N_0}{2}\left[1 + \frac{m^2}{2k_F q^3}\left[(\xi_q+\omega)^2 - 4\epsilon_F\xi_q\right]\log\left[\frac{\xi_q - v_F q + \omega}{\xi_q + v_F q + \omega}\right] + \frac{m^2}{2k_F q^3}\left[(\xi_q-\omega)^2 - 4\epsilon_F\xi_q\right]\log\left[\frac{\xi_q - v_F q - \omega}{\xi_q + v_F q - \omega}\right]\right]$$

Let us take a different approach — since we are interested in long wavelength collective modes, let us assume q is small.

In this case we can write the energies in denominator as

$$\varepsilon_{k+q/2} - \varepsilon_{k-q/2} = \frac{k^2 + k\cdot q + q^2/4}{2m} - \frac{k^2 - k\cdot q + q^2/4}{2m} = \frac{k\cdot q}{m}$$

As for the numerator:

$$n(\varepsilon_{k+q/2}) = n(\varepsilon_k) + \partial_\varepsilon n\left[\frac{1}{2m}k\cdot q + \frac{q^2}{8m}\right] + \frac{1}{2}\partial_\varepsilon^2 n\left[\frac{1}{2m}k\cdot q + \frac{q^2}{8m}\right]^2 + \dots \qquad \left[\frac{1}{2}2\right]2\frac{1}{2m}\frac{1}{8m}q^2\, k\cdot q$$

$$n(\varepsilon_{k-q/2}) = n(\varepsilon_k) + \partial_\varepsilon n\left[-\frac{1}{2m}k\cdot q + \frac{q^2}{8m}\right] + \frac{1}{2}\partial_\varepsilon^2 n\left[-\frac{1}{2m}k\cdot q + \frac{q^2}{8m}\right]^2 + \dots$$

$$n(\varepsilon_{k+q/2}) - n(\varepsilon_{k-q/2}) = \partial_\varepsilon n\,\frac{k\cdot q}{m} + \partial_\varepsilon^2 n\,\frac{q^2}{8m^2}k\cdot q + \dots$$

Putting these together

$$\chi_0 = \int dk\,\frac{1}{\omega - q\cdot k/m}\left[\partial_\varepsilon n\,\frac{k\cdot q}{m} + \partial_\varepsilon^2 n\,\frac{q^2}{8m^2}k\cdot q + \dots\right]$$

$$\chi_0^{(1)} = \int \frac{k^2 dk}{(2\pi)^3}\, 2\pi\int_{-1}^{1} dx\,\frac{1}{\omega - \frac{qkx}{m}}\,\partial_\varepsilon n\,\frac{qkx}{m} = \int \nu(\varepsilon)\,d\varepsilon\left[-\delta(\varepsilon-\varepsilon_f)\right]\int \frac{2\pi\,dx}{4\pi}\,\frac{qkx}{m\omega - qkx}$$

$\partial_\varepsilon n = -\delta(\varepsilon - \varepsilon_F)$

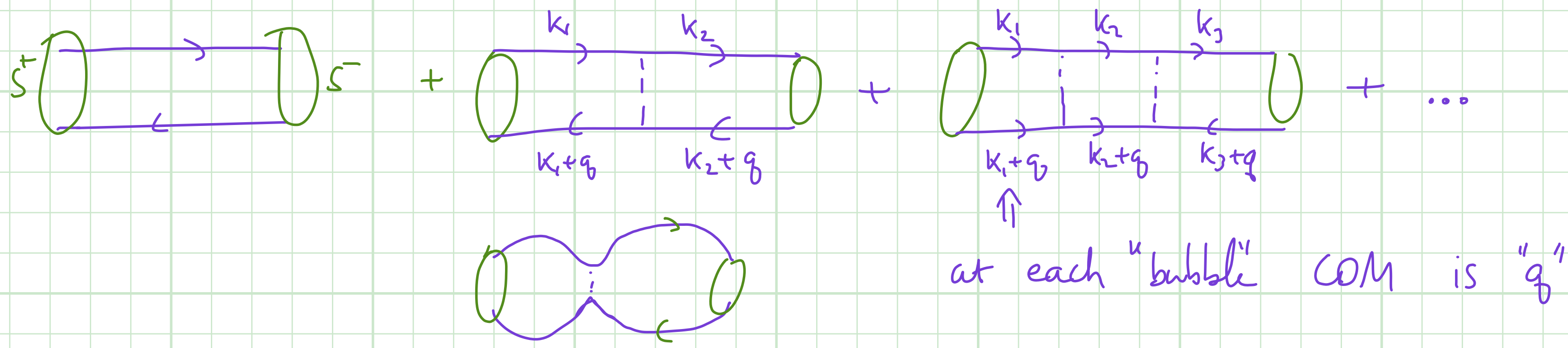
$x = \cos\theta$

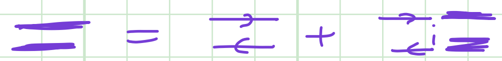
$$= -\frac{N_0}{2}\int_{-1}^{1} dx\,\frac{x}{\frac{m\omega}{qk_F} - x} = -\frac{N_0}{2}\left[-2 + z\log\left[\frac{z+1}{z-1}\right]\right] = N_0\left[1 + \frac{z}{2}\log\left(\frac{z-1}{z+1}\right)\right]$$

$\nu(\varepsilon_f)$

$z \equiv \frac{m\omega}{qk_F}$

Resum ...



$\Rightarrow$ 

$$\chi^{RPA}(q,\omega) = \chi^{(0)}(q,\omega) + \chi^{(0)}(q,\omega)\, U\, \chi^{RPA}(q,\omega)$$

$$\chi^{RPA}(q,\omega) = \frac{\chi^{(0)}(q,\omega)}{1-\chi^{(0)}(q,\omega)\, U}$$

Pole structure:

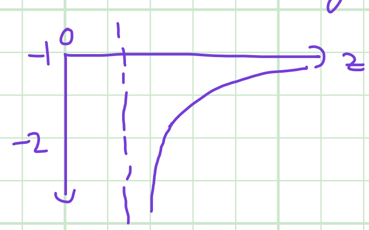
$$1 - N_0 U\left[1 + \frac{z}{2}\log\left(\frac{z-1}{z+1}\right)\right] = 0$$

$$\frac{1}{N_0 U} - 1 = \frac{z}{2}\log\left(\frac{z-1}{z+1}\right) \qquad (*)$$

Solve graphically:

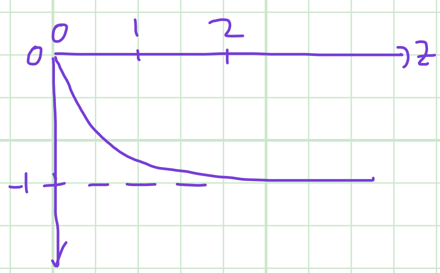
The RHS is completely Real in only two cases:

$z \in \mathbb{R} \Rightarrow$ RHS



RHS $\in \{-\infty, -1\}$

$z \in i\mathbb{R} \Rightarrow$ RHS



RHS $\in \{-1, 0\}$

Hence all solutions have the RHS as negative.
Let's plot the LHS and find the "propagating" collective spin-modes